

2.2.6 Arc Length of a Curve

► **Rule 1**: If $f : [a, b] \rightarrow \mathbb{R}$ has continuous derivative on $[a, b] \Rightarrow$ Length of the curve is $\int_a^b \sqrt{1 + [f'(x)]^2} dx$.

► **Rule 2**: The length of the curve $y = f(x)$ within $x = a$ to $x = b$ is $\int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx$.

► **Rule 3**: The length of the curve $x = g(y)$ within $y = c$ to $y = d$ is $\int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2} dy$.

► **Rule 4**: The length of the parametric curve $x = f_1(t)$, $y = f_2(t)$ within $t = t_1$ to $t = t_2$ is $\int_{t_1}^{t_2} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt$.

► **Rule 5**: The length of the polar curve $r = f(\theta)$ within $\theta = \theta_1$ to $\theta = \theta_2$ is $\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} d\theta$.

[Do It Yourself] 2.30. The length of the curve $y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 7$ from $x = 1$ to $x = 8$ equals (A) 99/8. (B) 117/8. (C) 99/4. (D) 117/4.

[Hint : Easy integration]

Example 2.2. Find the perimeter of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

\Rightarrow The parametric form: $x = a \sin^3 t$, $y = a \cos^3 t$.

Now in first quadrant $x = 0, a \Rightarrow t = 0, \pi/2$.

Length of perimeter in first quadrant is $\int_0^{\pi/2} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt = 3a \int_0^{\pi/2} \sin t \cos t dt = \frac{3a}{2}$.

So the total length is $4 \cdot \frac{3a}{2} = 6a$.

Example 2.3. Find the length of the loop of the curve $x = t^2$, $y = t - \frac{t^3}{3}$.

\Rightarrow We first try to draw the curve. $\left(\begin{array}{ccc|ccc|ccc} t & x & y & t & x & y & t & x & y \\ \hline 0 & 0 & 0 & 2 & 4 & 2/3 & -\sqrt{3} & 3 & 0 \\ 1 & 1 & 2/3 & -2 & 4 & -2/3 & \sqrt{2} & 2 & \sqrt{2}/3 \\ -1 & 1 & -2/3 & \sqrt{3} & 3 & 0 & -\sqrt{2} & 2 & -\sqrt{2}/3 \end{array} \right)$

We can easily draw the graph and see that loop is on x -axis from $x = 0$ to $x = 3$ i.e. $t = 0$ to $t = \sqrt{3}$. It is both side of x -axis.

Length of loop = $2 \int_0^{\sqrt{3}} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt = 2 \int_0^{\sqrt{3}} \sqrt{4t^2 + (1 - t^2)^2} dt = 4\sqrt{3}$.

Example 2.4. Show that the length of one arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ is $8a$.

\Rightarrow Draw the curve. Now $y = 0 \Rightarrow \cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi, \dots$.

So the required length is $\int_0^{2\pi} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt = \int_0^{2\pi} \sqrt{[a(1 - \cos t)]^2 + [a \sin t]^2} dt = 8a$.

Example 2.5. Find the length of the perimeter of the cardioid $r = a(1 - \cos \theta)$ and show that the arc of the upper half of the curve is bisected by $\theta = 2\pi/3$.

\Rightarrow Draw the curve. Now for upper half $\theta = 0$ to $\theta = \pi$.

So the total length is $2 \int_0^\pi \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} d\theta = 2 \int_0^\pi \sqrt{[a(1 - \cos t)]^2 + [a \sin t]^2} dt = 8a$.

□ Length of upper half is $4a$. Therefore, we have to show: $\int_0^{2\pi/3} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} d\theta = 2a$.

Now $\int_0^{2\pi/3} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} d\theta = \int_0^{2\pi/3} \sqrt{[a(1 - \cos t)]^2 + [a \sin t]^2} dt = 2a$.

[Do It Yourself] 2.31. A tangent is drawn on the curve $y = \frac{1}{3}\sqrt{x^3}$, ($x > 0$) at the point $P(1, \frac{1}{3})$ which meets the x -axis at Q . Then find the length of the closed curve $OQPO$, where O is the origin. [Hint: Curve + two lines, Easy]

[Do It Yourself] 2.32. Consider a differentiable function f on $[0, 1]$ with the derivative $f'(x) = 2\sqrt{2x}$. Find the arc length of the curve $y = f(x)$, $0 \leq x \leq 1$. [Hint: Easy]

[Do It Yourself] 2.33. Find the length of the curve $y = \sqrt{4 - x^2}$ from $x = -\sqrt{2}$ to $x = \sqrt{2}$. [Hint: Easy]

[Do It Yourself] 2.35. Let $f : [0, \infty) \rightarrow [0, \infty)$ be twice differentiable and increasing function with $f(0) = 0$. Suppose that, for any $t \leq 0$, the length of the arc of the curve $y = f(x)$, $x \leq 0$ between $x = 0$ and $x = t$ is $\frac{2}{3}[(1 + t)^{\frac{3}{2}} - 1]$. Then $f(4)$ is equal to (A) $11/3$. (B) $13/3$. (C) $14/3$. (D) $16/3$.

[Hint: Use $\int_0^t \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx = \frac{2}{3}[(1 + t)^{\frac{3}{2}} - 1]$, and find $f(t)$]

[Do It Yourself] 2.36. Show that the length of the hypercycloid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ is $\frac{4(a^2 + ab + b^2)}{a + b}$.

[Do It Yourself] 2.37. Show that the length of $r = a \cos^3(\frac{\theta}{3})$ is $\frac{3\pi a}{2}$.

[Do It Yourself] 2.38. Show that the length of the loop of $x = t^2$, $y = t - \frac{t^3}{3}$ is $4\sqrt{3}$.

[Do It Yourself] 2.39. Show that the length of the loop of $9ay^2 = (x - 2a)(x - 5a)^2$ is $4\sqrt{3}a$.

2.2.7 Volumes of Solids of Revolution

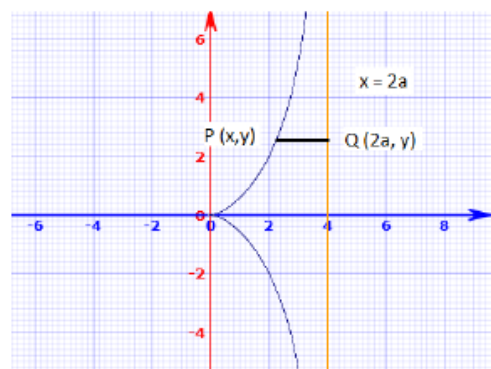
► **Rotation around x -axis**: The volume V generated by revolving the curve $y = f(x)$ about the x -axis and bounded by $x = a$ to $x = b$ is $V = \pi \int_a^b y^2 dx$.

► **Rotation around y -axis**: The volume V generated by revolving the curve $x = g(y)$ about the y -axis and bounded by $y = c$ to $y = d$ is $V = \pi \int_c^d x^2 dy$.

Example 2.6. Find the volume of the solid obtained by the revolution of the Cissoid $y^2(2a - x) = x^3$ about its asymptote.

⇒ The Cissoid $y^2(2a - x) = x^3$ has asymptote $x = 2a$ (In graph we take $a = 2$).

Now the volume of the solid generated by $y^2(2a - x) = x^3$ around $x = 2a$ has two parts. One is upper side (first quadrant) and another one is lower side (fourth quadrant).



Let, $P(x, y)$ be any point on the curve.

So, $Q(2a, y)$ is the point on $x = 2a$.

It implies $PQ = 2a - x$.

Suppose $x = 2a$ cuts x -axis at R .

So $RQ = \sqrt{\frac{x^3}{2a-x}} \Rightarrow d(RQ) = \frac{\sqrt{x}(3a-x)}{(2a-x)^{3/2}} dx$.

So the upper volume is $\pi \int_0^{2a} (PQ)^2 d(RQ)$.

The required volume $V = 2\pi \int_0^{2a} (PQ)^2 d(RQ)$.

Therefore, $V = 2\pi \int_0^{2a} (2a - x)^2 \frac{\sqrt{x}(3a-x)}{(2a-x)^{3/2}} dx = 2\pi \int_0^{2a} \sqrt{x(2a-x)}(3a-x) dx$.

Let $x = 2a \sin^2 \theta \Rightarrow dx = 2a \sin 2\theta d\theta$. Therefore,

$$V = 2\pi \int_0^{\pi/2} \sqrt{x(2a-x)}(3a-x) dx = 4a^3 \pi \int_0^{\pi/2} \sin^2 2\theta (2 + \cos 2\theta) d\theta = 4a^3 \pi (\pi/2) = 2a^3 \pi^2.$$

[Do It Yourself] 2.42. Find the volume of the solid generated by revolving the region bounded by the parabola $x = 2y^2 + 4$ and the line $x = 6$ about the line $x = 6$.

(A) $78\pi/15$. (B) $91\pi/15$. (C) $64\pi/15$. (D) $117\pi/15$.

[Hint: Draw curve. Here $V = 2\pi \int_4^6 (6-x)^2 \frac{1}{2\sqrt{2}\sqrt{x-4}} dx$]

[Do It Yourself] 2.44. Find the volume of the solid formed by revolving the curve $y = x$ between $x = 0$ and $x = 1$ about the x -axis. [Hint: Easy]

[Do It Yourself] 2.45. The volume of the solid of revolution generated by revolving the area bounded by the curve $y = \sqrt{x}$ and the straight lines $x = 4$ and $y = 0$ about the x -axis, is

(A) 2π . (B) 4π . (C) 8π . (D) 12π .

Example 2.7. A region bounded by $y^2 = x$ and $x^2 = y$ rotates about x -axis. Find the volume of the solid of revolution.

\Rightarrow We can easily draw the curves and see that the required volume $V = V_1 - V_2$.

Here V_1 : Volume of the solid by the revolution of $y^2 = x$, $x = 0$, $x = 1$.

V_2 : Volume of the solid by the revolution of $x^2 = y$, $x = 0$, $x = 1$.

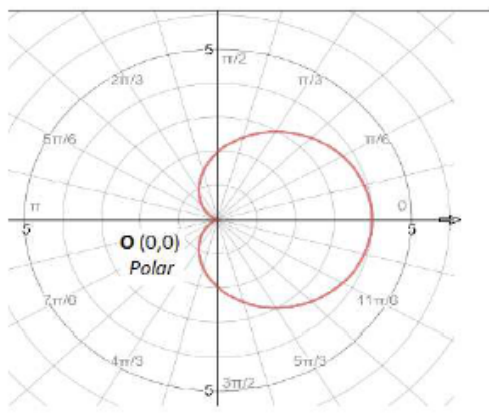
Therefore, $V = \pi \int_0^1 x dx - \pi \int_0^1 x^4 dx = \pi(\frac{1}{2} - \frac{1}{5}) = \frac{3\pi}{10}$.

[Do It Yourself] 2.47. Find the volume of the solid obtained by rotating the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about x -axis.

[Hint: Draw the curve and check $V = \pi \int_{-a}^a y^2 dx = \pi \int_{-a}^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^3 dx = \frac{32\pi a^3}{105}$.]

Example 2.9. Show that the volume of the solid obtained by the cardioid $r = a(1 + \cos \theta)$ about about the initial line is $\frac{8\pi a^3}{3}$.

\Rightarrow We will transform polar (r, θ) to cartesian (x, y) system. [In plot $a = 2$]



Now the pole is $O (r = 0, \theta = 0)$.

\overrightarrow{OX} is the initial line.

Rotation: $0 \leq r \leq 2a$, $0 \leq \theta < \pi$.

Now $x = r \cos \theta$, $y = r \sin \theta$.

Curve: $x = a(1 + \cos \theta) \cos \theta$,

$y = a(1 + \cos \theta) \sin \theta$.

$dx = -a \sin \theta (1 + 2 \cos \theta) d\theta$.

The required volume $V = \pi \int y^2 dx$.

Now we have to find the range of x .

At $\theta = 0$, $x = 2a$ and $\theta = \pi$, $x = 0$.

The required volume $V = \pi \int_0^{2a} y^2 dx = -\pi \int_{\pi}^0 a^2 (1 + \cos \theta)^2 a \sin^3 \theta (1 + 2 \cos \theta) d\theta = \pi a^3 \int_0^{\pi} (1 + \cos \theta)^2 \sin^3 \theta (1 + 2 \cos \theta) d\theta = \frac{8\pi a^3}{3}$. [take $z = \cos \theta$]

[Do It Yourself] 2.49. Show that the volume of a sphere with radius a is $\frac{4}{3}\pi a^3$.

[Do It Yourself] 2.50. Find the volume generated by the following curve rotating around x -axis:

i) $y = \cos x$ between $x = 0$, $\pi/2$. ii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ between $x = 0$, $y = 0$. iii) The area between $9y = 4(9 - x^2)$ and $4x + 3y = 12$. iv) Loop of the curve $x = t^2$, $y = t - \frac{t^3}{3}$.

[Ans: $\frac{\pi^2}{4}$, $\frac{8\pi a^3}{15}$, $\frac{48\pi}{5}$, $\frac{3\pi}{4}$]

[Do It Yourself] 2.51. Show that the volume of the solid obtained by the cardioid $r = a(1 - \cos \theta)$ about about the initial line is $\frac{8\pi a^3}{3}$.

2.2.8 Surface of Revolution

► **Rotation around x - axis**: The surface area S generated by revolving the curve $y =$

$f(x)$ about the x - axis and bounded by $x = a$ to $x = b$ is $S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$.

► **Rotation around y - axis**: The volume S generated by revolving the curve $x = g(y)$

about the y - axis and bounded by $y = c$ to $y = d$ is $S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$.

► Finding surface of revolution is similar to finding the **Volumes of Solids of Revolution**.

[Do It Yourself] 2.55. Find the area of the solid obtained by rotating the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about x - axis.

[Hint: $S = 2\pi \int_{-a}^a y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_{-\pi/2}^{\pi/2} a \cos^3 t \sqrt{1 + \frac{\sin^2 t}{\cos^2 t}} 3a \sin^2 t \cos t dt = \frac{8\pi a^2}{5}$]

[Do It Yourself] 2.56. Show that the surface area of the solid obtained by rotating the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ about x - axis is $\frac{64\pi a^2}{3}$.